Determination of the Direction That Has Maximum Phase-Velocity for the 2-D and 3-D FDTD Methods Based on Yee's Algorithm

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***Abstract*—To easily evaluate the performance of the numerical dispersion of the FDTD method, a simple and efficient manner to theoretically determine the special direction that has the maximum phase-velocity for arbitrarily shaped two-dimensional (2-D) and three-dimensional (3-D) Yee's cells is proposed. It is found that the wave propagating along the special direction obeys the following physical rule: the time that the wave takes to traverse the spatial step sizes along each of the main axes of the cell is identical. It is demonstrated that this special direction is closely related to the aspect ratio(s) of the cells, rather than the diagonals of the cells. Moreover, the mathematical expressions for the maximum phase-velocity of the 2-D and 3-D cells with arbitrary aspect ratios are derived for the first time.**

***Index Terms*—FDTD method, maximum phase-velocity, numerical dispersion.**

I. Introduction

IT HAS been shown [1] that the performance of the numerical dispersion of the finite-difference time-domain (FDTD) method can be quantified by two normalized error measures: i) the physical phase-velocity error $\Delta\nu_{physical}$ and ii) the velocity-anisotropy error $\Delta\nu_{aniso}$. They are defined as [1]

$$\Delta\nu_{physical} = \left(1 - \frac{\nu_{\min}}{c}\right) \times 100\% \tag{1.1}$$

$$\Delta\nu_{aniso} = \frac{\nu_{\max} - \nu_{\min}}{\nu_{\min}} \times 100\%. \tag{1.2}$$

From (1) one can see that in order to evaluate the performance of the numerical dispersion $\nu_{\min}$ and $\nu_{\max}$ must be known. Although for both the two-dimensional (2-D) and three-dimensional (3-D) FDTD methods based on Yee's scheme $\nu_{\min}$ and $\nu_{\max}$ can be numerically obtained by solving the corresponding numerical dispersion relations, it will be more convenient if the analytical solutions are available. Deriving the mathematical expression for $\nu_{\min}$ can be easily done as $\nu_{\min}$ appears along the axis that has the biggest spatial increment [1]. In the past, the mathematical expression for $\nu_{\max}$ was derived only for the cases where Yee's cells are either square (for 2-D case) or cubic (for 3-D case) since in these cases the special direction that has the maximum phase-velocity is the same as the diagonals of the square or cubic cell [1], [2]. For nonsquare and noncubic Yee's cells, however, mathematical expression for $\nu_{\max}$ was not found. This is because for the nonsquare and noncubic cells the special direction is no longer identical to the diagonal of the cells; and most importantly, it was unclear that for the wave propagating along the special direction what physical rule should be obeyed.

In this paper, we first find the physical rule that is obeyed by the wave propagating along the special direction. Consequently, the exact angles that define the special direction are determined for both general 2-D and 3-D Yee's cells. It is found that although in general the special direction is not directly associated with the diagonal of the 2-D and 3-D Yee's cells, it closely relates to the shape of the cells. In addition, the mathematical expressions of $\nu_{\max}$ are derived for the 2-D and 3-D cells with arbitrary aspect ratios.

II. Determination of the Special Direction for 2-D and 3-D Yee's Cells

Two-Dimensional Case: If for an arbitrary 2-D Yee's cell Δx and Δy are the cell sizes in the x and y directions, respectively, then the numerical dispersion relation of the 2-D FDTD method is

$$\frac{1}{(\Delta x)^2}\sin^2\left(\frac{k_x\Delta x}{2}\right) + \frac{1}{(\Delta y)^2}\sin^2\left(\frac{k_y\Delta y}{2}\right) = \frac{1}{(c\Delta t)^2}\sin^2\left(\frac{\omega\Delta t}{2}\right) \tag{2}$$

and the time step used in the 2-D FDTD method is limited by

$$\Delta t \le \Delta t_{\max}^{2D} = \frac{1}{c\sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}}} \tag{3}$$

where $c = 1/\sqrt{\varepsilon\mu}$ is the speed of light in the medium; ω is the wave angle frequency; and k_x and k_y are the numerical wavenumbers in the x and y directions, respectively. If we assume that the wave propagates at an angle ϕ with respect to the positive x direction, then the numerical wavenumbers in the x and y directions are given by

$$k_x = k\cos\phi, \quad \text{and} \quad k_y = k\sin\phi. \tag{4}$$

According to the definition of the special direction, one can see that when the wave propagates along this special direction the following physical rule must be obeyed: the time that the wave takes to traverse the spatial step sizes along the x and y directions must be identical. This is the only condition that ensures the wave propagating along this special direction have

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the maximum phase-velocity. The amount of the traveling time of the wave spends in the x and y directions of the 2-D cell can be represented by $\Delta t_x (=\Delta x/\nu_x)$ and $\Delta t_y (=\Delta y/\nu_y)$, respectively; where $\nu_x (=\omega/k_x)$ and $\nu_y (=\omega/k_y)$ are the phase-velocity of the wave along the x and y directions. Therefore, while the physical rule $\Delta t_x = \Delta t_y$ is satisfied, one has

$$k_x \Delta x = k_y \Delta y. \tag{5}$$

Combining (4) with (5), one can obtain the exact angle of the special direction for a general 2-D Yee's cell

$$\phi = \tan^{-1}(R) = \tan^{-1}\left(\frac{\Delta x}{\Delta y}\right) \tag{6}$$

where $R = \Delta x/\Delta y$ is the aspect ratio of the 2-D Yee's cell. Note that the special direction is identical to the diagonal of a 2-D cell only when the 2-D cell is square (i.e., $R = 1.0$).

Substituting (4) and (6) into (2), one can obtain the mathematical expression for the numerical wavenumber k along the special direction of the 2-D FDTD method

$$k = \frac{2\sqrt{1+R^2}}{\Delta x} \sin^{-1}\left[\frac{1}{S}\sin\left(\frac{\omega \Delta x S}{2c\sqrt{1+R^2}}\right)\right] \tag{7}$$

where $S = \Delta t/\Delta t_{\max}^{2D}$ is the Courant number of the 2-D FDTD method. Hence, the mathematical expression of $\nu_{\max}$ along the special direction of the 2-D FDTD method is

$$\nu_{\max} = \frac{\omega}{k} = \frac{\omega \Delta x}{2\sqrt{1+R^2}\sin^{-1}\left[\frac{1}{S}\,\sin\left(\frac{\omega\Delta x S}{2c\sqrt{1+R^2}}\right)\right]}. \tag{8}$$

From (8), one can see that $\nu_{\max} = c$ if $S = 1$. This also means that the numerical wavenumber k along the special direction is identical to the theoretical wavenumber ω/c if the maximum time step is used.

Three-Dimensional Case: If for an arbitrary 3-D Yee's cell Δx, Δy, and Δz are the cell sizes in the x, y, and z directions, respectively, then the numerical dispersion relation of the 3-D FDTD method is

$$\frac{1}{(\Delta x)^2}\,\sin^2\left(\frac{k_x\Delta x}{2}\right) + \frac{1}{(\Delta y)^2}\,\sin^2\left(\frac{k_y\Delta y}{2}\right) + \frac{1}{(\Delta z)^2}\,\sin^2\left(\frac{k_z\Delta z}{2}\right) = \frac{1}{(c\Delta t)^2}\,\sin^2\left(\frac{\omega\Delta t}{2}\right) \tag{9}$$

and the time step used in the 3-D FDTD method is limited by

$$\Delta t \le \Delta t_{\max}^{3D} = \frac{1}{c\sqrt{\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2}}}. \tag{10}$$

Assume that ϕ and θ are the azimuth and polar angles of the wave propagation direction, then the numerical wavenumbers in the x, y, and z directions are

$$k_x = k\sin\theta\cos\phi, \quad k_y = k\sin\theta\sin\phi, \quad \text{and} \quad k_z = k\cos\theta. \tag{11}$$

If the wave propagates exactly along the special direction of the 3-D Yee's cell, then the time that the wave takes to traverse the spatial step sizes along the x, y, and z directions of the 3-D cell must also be identical. Therefore, similar to the 2-D case, the following condition must be hold for the wave propagating along the special direction of the 3-D Yee's cell:

$$k_x\Delta x = k_y \Delta y = k_z \Delta z \tag{12}$$

Combining (11) with (12), one can find the exact angles of the special direction for a general 3-D Yee's cell

$$\phi = \tan^{-1}(R_y), \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{\sqrt{1+R_y^2}}{R_z}\right) \tag{13}$$

where $R_y = \Delta x/\Delta y$ and $R_z = \Delta x/\Delta z$ are the aspect ratios of the 3-D Yee's cell. Note that the special direction is identical to the diagonal of a 3-D cell only when the 3-D cell is cubic (i.e., $R_y = R_z = 1.0$).

Substituting (11) and (13) into (9), one can obtain the mathematical expression for the numerical wavenumber k along the special direction of the 3-D FDTD method

$$k = \frac{2\sqrt{1+R_y^2+R_z^2}}{\Delta x}\,\sin^{-1}\left[\frac{1}{S}\,\sin\left(\frac{\omega\Delta x S}{2c\sqrt{1+R_y^2+R_z^2}}\right)\right] \tag{14}$$

where $S = \Delta t/\Delta t_{\max}^{3D}$ is the Courant number of the 3-D FDTD method. Thus, the mathematical expression of $\nu_{\max}$ along the special direction of the 3-D FDTD method is

$$\nu_{\max} = \frac{\omega}{k} = \frac{\omega\Delta x}{2\sqrt{1+R_y^2+R_z^2}\sin^{-1}\left[\frac{1}{S}\,\sin\left(\frac{\omega\Delta x S}{2c\sqrt{1+R_y^2+R_z^2}}\right)\right]}. \tag{15}$$

From (15), one can see that, similar to the 2-D case, we still have $\nu_{\max} = c$ if $S = 1$. This implies that for the 3-D case the numerical wavenumber k along the special direction is also identical to the theoretical wavenumber ω/c if the maximum time step is adopted.

III. Numerical Validations

The numerical dispersion relations (2) and (9) can be numerically solved with a root-finding technique. This implies that the angles of the special direction can be numerically determined. Therefore, comparing with the numerical solutions can validate the theory proposed in Section II for finding the exact angles of the special directions.

Two-dimensional case: Without loss of generality, we can assume that for the 2-D cell we have $\Delta x \ge \mathrm{Max}(\Delta x, \Delta y)$. In addition, we define $N = \lambda/\Delta x$ as the mesh resolution of the 2-D cell. Fig. 1 shows the normalized phase-velocity (v_p/c) [obtained numerically from (2)] as functions of R, S, and ϕ. In particular, the values of R are chosen to be 1.0, 1.5, and 2.0; and the values of S are 1.0 or 0.5. In addition, the mesh resolution $N = 20$ is used for all the cases. From Fig. 1 one can see that, as expected, the special direction that has the maximum phase-velocity varies with R. To ensure the accuracy of the angle obtained numerically from (2), a small increment $\Delta\phi = 0.0001°$

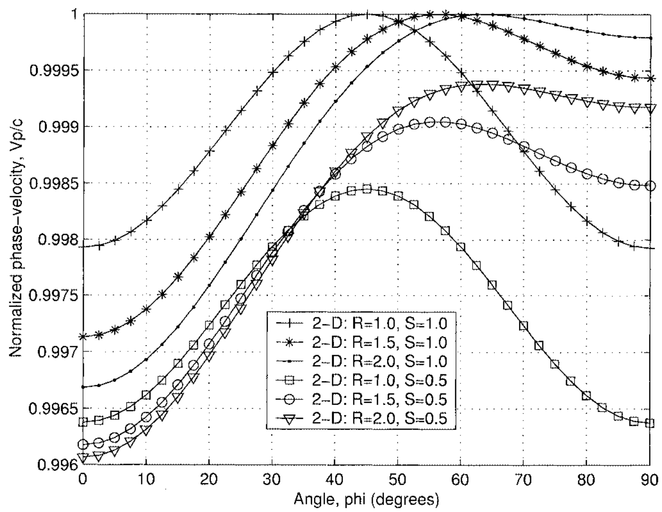

Fig. 1. Normalized phase-velocity (v_p/c) as functions of R and S for the 2-D FDTD method.

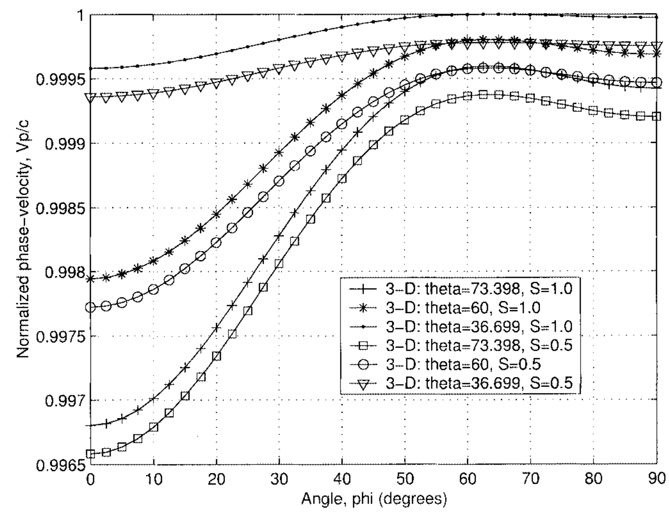

Fig. 2. Normalized phase-velocity (v_p/c) as functions of θ and S for the 3-D FDTD method.

is used. For $R = 1.0$, 1.5, and 2.0 the angle ϕ (of the special direction) obtained numerically from (2) is 45°, 56.31°, and 63.435°, respectively. These angles are just the same as those obtained directly from (6). Note that for the 2-D cells with $R = 1.0$, 1.5, and 2.0 the angles of their diagonals are defined by $\phi_0 = 45°$, 33.69°, and 26.565°, respectively. This means that only for the square cell (i.e., $R = 1.0$) the special direction is identical to the diagonal of the cell. In addition, from Fig. 1 one can see that for $S = 1.0$ or 0.5 the angle for the special direction is unchanged. Moreover, the values of $\nu_{\max}$ obtained numerically from (2) are also the same as those obtained from (8).

Three-dimensional case: Without loss of generality, we can assume that for the 3-D cell we have $\Delta x \geq \mathrm{Max}(\Delta x, \Delta y, \Delta z)$, and define $N = \lambda/\Delta x$ as the mesh resolution of the 3-D cell. As an example, we use $R_y = 2.0$, $R_z = 3.0$, and $N = 20$ for validation. Fig. 2 shows the normalized phase-velocity (v_p/c) [obtained numerically from (9)] as functions of θ, ϕ and S. To ensure the accuracy of the angles obtained numerically from (9), a small increment $\Delta\phi = \Delta\theta = 0.0001°$ is used. Note that the angles $\theta = 73.398°$ and 36.699° shown in Fig. 2 are the polar angles of the diagonal and the special direction of the 3-D cell, respectively. The angles of the special direction obtained numerically from (9) are $\phi = 63.435°$ and $\theta = 36.699°$, which are the same as those obtained directly from (13). In addition, the angles of the diagonal of the 3-D cell are $\phi_0 = 26.565°$ and $\theta_0 = 73.398°$, which implies that for the 3-D cell under consideration the special direction is different from its diagonal [however, if the cubic cell (i.e., $R_y = R_z = 1.0$) is used, then the special direction will be identical to the diagonal of the cell]. In addition, the values of $\nu_{\max}$ obtained numerically from (9) are also the same as those obtained directly from (15). More studies indicate that the above conclusion is also true for the cases where arbitrary values are used for R_y and R_z. These investigations certainly confirm the validity of the theory proposed in this letter.

IV. Conclusions

Based on the physical rule discovered in this paper, the exact angles of the special direction are determined for both general 2-D and 3-D Yee's cells. In addition, the mathematical expressions for the maximum phase-velocity are also derived. This, in turn, makes the evaluation of the performance of the numerical dispersion of the FDTD method based on Yee's cells with arbitrary aspect ratios simpler and easier. Furthermore, from the analysis carried out in this paper one can know that the special direction is generally not the same as the diagonal of the Yee's cells; and it reduces to the diagonal *only* when the Yee's cells are either square (in 2-D case) or cubic (in 3-D case).

References

[1] A. Taflove and S. Hagness, *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, 2nd ed. Boston, MA: Artech House, 2000.

[2] J. B. Schneider and R. J. Kruhlak, "Dispersion of homogeneous and inhomogeneous waves in the Yee finite-difference time-domain grid," *IEEE Microwave Theory Tech.*, vol. 49, pp. 280–287, Feb. 2001.